Noise-enhanced Nonlinear PDE for Edge Restoration in Scalar Images

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***Abstract*—The report proposed an interpretation for the mechanism of noise-enhanced image restoration with nonlinear PDE (Partial Differential Equation) recently demonstrated in literature. A link is established between the action of noise in a nonlinear Perona–Malik anisotropic diffusion and stochastic resonance in memoryless nonlinear systems for 1-D signals.**

***Keywords*-stochastic resonance, anisotropic diffusion, image restoration, nonlinear image processing.**

I. Introduction

It is progressively realized that noise can play a constructive role in nonlinear information processes. The starting point of the investigation of such useful-noise effect has been the study of stochastic resonance [Benzi et al., 1981], [Benzi et al., 1982], [Wiesenfeld and Moss, 1995]. Originally introduced to describe the mechanism of a constructive action of a white Gaussian noise in the transmission of a sinusoid by a nonlinear dynamic system governed by a double-well potential [Gammaitoni et al., 1989], [McNamara and Wiesenfeld, 1989], the phenomenon of stochastic resonance has experienced large varieties of extensions with variations concerning the type of noise, the type of information-carrying signal or the type of nonlinear system interacting with the signal-noise mixture (see for example [Gammaitoni et al., 1998] for a review in physics, [Harmer et al., 2002] for an overview in electrical engineering and [Chapeau-Blondeau and Rousseau, 2002] for the domain of signal processing). All these extensions of the original setup preserve the possibility of improving the processing of a signal by means of an increase in the level of the noise coupled to this signal. New forms of useful-noise effect, related to stochastic resonance or not, continue to be demonstrated. One current domain of interest is the study of non-trivial transposition of stochastic resonance to image processing [Blanchard et al., 2008], [Rousseau et al., 2010]. In this report, we come back on the results of a prospective study presented in [Histace and Rousseau, 2006] and we establish a link between this work [Histace and Rousseau, 2006] and the mechanisms encountered in the monodimensional domain.

This article is organized as follows: we first recall the stochastic anisotropic diffusion equation of interest originally proposed in [Histace and Rousseau, 2006]. We then propose a simplified version of this equation stochastic diffusion equation preserving the essential properties of the former historical equation proposed in [Perona and Malik, 1990]. Therefrom, we establish a formal analogy between useful-noise effect in image anisotropic diffusion and the mechanism of stochastic resonance in static non-linearity with additive signal–noise mixture in monodimensional signals.

II. Stochastic anisotropic diffusion

The original Perona-Malik process [Perona and Malik, 1990] is an anisotropic diffusion process inspired from the physics of temperature diffusion in which an observable noisy image ψ_0 is restored by considering the solution of the PDE given by

$$\frac{\partial \psi}{\partial t} = div(g(\|\nabla\psi\|)\nabla\psi), \quad \psi(x,y,t=0)=\psi_0 \qquad (1)$$

where the anisotropy of this diffusion process is governed by $g(\cdot)$ a nonlinear decreasing function of the norm of the gradient $\nabla\psi$. In this study, we consider the process given by

$$\frac{\partial \psi}{\partial t} = div(g_\eta(\|\nabla\psi\|)\nabla\psi) \ , \qquad (2)$$

which is of a form similar to Eq. (1) except for the nonlinear function $g_\eta(\cdot)$ which is given by

$$g_\eta(u) = g(u+\eta(x,y)) \ , \qquad (3)$$

where η is a noise assumed independent and identically distributed with probability density function (pdf) $f_\eta(u)$ and rms amplitude σ_η. The noise η, which is distinct from the native noise component ξ to be removed, is a purposely added noise applied to influence the operation of $g(\cdot)$. In [Histace and Rousseau, 2006], we have shown that the injection of a Gaussian noise in Eq. (3) can improve the restoration process by comparison with standard Perona-Malik process of Eq. (1) when the native noise component ξ is a Gaussian, impulsive or multiplicative noise and with $g(\cdot)$ given by

$$g(u) = e^{-\frac{\|u\|^2}{k^2}} \ . \qquad (4)$$

In this expression, parameter k can be seen as a soft threshold controlling the decrease of $g(\cdot)$ and the amplitude of the gradients to be preserved from the diffusion process. Our previous works [Histace and Rousseau, 2006] and [Histace and Rousseau, 2007] has shown, as a proof of feasability, that an injection of a non zero amount of noise could help the restoration process when the threshold k is ill-positioned. We propose here to investigate the inner mechanism of the useful-noise effect shown in [Histace and Rousseau, 2006] and [Histace and Rousseau, 2007]. To this purpose, we propose to simplify the nonlinear function $g(\cdot)$ of Eq. (4). The diffusive function of Eq. (4) was chosen in [Histace and Rousseau, 2006] because it corresponds to the historical function proposed in [Perona and Malik, 1990]. This choice nevertheless presents some drawbacks for the complete understanding of the useful-noise effect since the presence in the analytical definition of $g(\cdot)$ function of a L_2 norm of the purposely noised gradient of the image leads to an offset shifting that makes the interpretation of the impact of the noise uneasy.

In this report, we choose to simplify the shape of $g(\cdot)$ into a hard threshold non-linearity given by

$$g(s) = \left\{ \begin{array}{l} 1 \text{ if } s \geq k \\ 0 \text{ if } s < k \end{array} \right. , \quad (5)$$

where parameter k is now a hard threshold. This function integrates a hard non-linearity in order to set in a binary way the diffusion threshold. Moreover, this non-linearity is only function of gradient itself in order to only emphasis the effect of the purposely injection of noise and to avoid the shifting effect described above. One can note that despite this methodological choice regarding $g(\cdot)$ function, this latter is just a simplified version of the former function proposed in [Perona and Malik, 1990] and still embed the fundamental elements of the classical anisotropic diffusion.

For illustration, the data to be restored is also chosen in its most simplest form. We consider a monodimmensional signal ψ_{ori} taken as a unit step function modeling an edge within a noisy image. ψ_0 will denote the noisy version of ψ_{ori}. The goal is now to restore the noisy step version without altering the hard discontinuity of ψ_{ori}. More, we want to show that injection of noise within the restoration process can lead to overpass the classical weak point of Perona-Malik process: a lack of robustness regarding k parameter illustrated Fig. 2. As visible in Eq. (5) and Fig. 2, parameter k is a very sensitive parameter for the restoration task since little variations of k can lead to completely different restoration results [1]. One can notice on Fig. 2 that for $k < 0.5$ ψ_{ori} is not altered by the Perona-Malik diffusion process and that corrupting noise ξ is removed, whereas for $k > 0.5$ if corrupting noise ξ is still removed, a smoothing of ψ_{ori} is also introduced. This translates in two dimension as an alteration of boundaries within images for a bad tuning of parameter k. In [Histace and Rousseau, 2006] we have shown that the stochastic variant of Perona-Malik process of Eq. (2) has a stronger robustness toward the tuning of parameter k. We provide an interpretation of the mechanism for this useful-noise effect.

[1]For this experiment, to apply classical Perona-Malik process to ψ_{ori}, corresponding equation is sampled with a time step τ such as $t_n = n\tau$ where n is the number of iterations in the process and t_n the corresponding scale

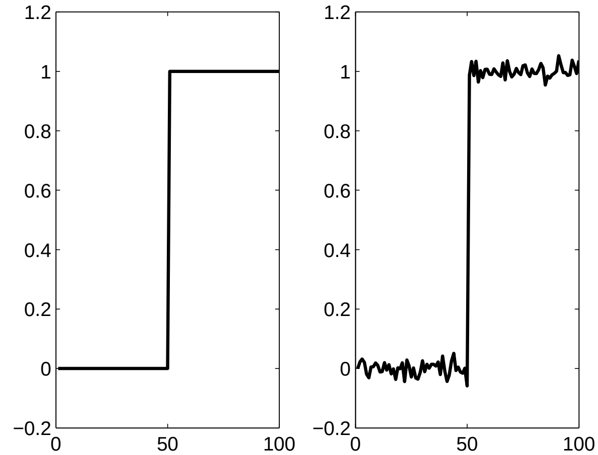

Figure 1. Illustration of the monodimensional function used for the study. On the left, the original ψ_{ori} function. On the right, the corrupted version ψ_0 (ξ is chosen gaussian).

III. Stochastic restoration: theoretical study

A. Preliminary calculations

The non-linearity of Eq. (2) can be classified as a static or memoryless non-linearity. Possibility of useful-noise effect in static non-linearity has been intensively studied (see [Chapeau-Blondeau and Rousseau, 2002] for a review). The action of the additive noise $\eta(x,y)$ can be understood as a shaping by noise of the input–output characteristic which on average becomes equivalent to

$$g_{eff}(s) = E[g(s+\eta(x,y))] = \int_{-\infty}^{+\infty} g(u) f_\eta(u-s) du \ , \quad (6)$$

with $f_\eta(u)$ the probability density function of the purposely injected noise η. In the case of the hard quantizer of Eq. (5) with threshold k, Eq. (6) becomes

$$g_{eff}(s) = F_\eta(k-s) \ , \quad (7)$$

where $F_\eta(.)$ is the cumulative distribution function of the probability density function of $f_\eta(u)$. If we consider the case where $f_\eta(u)$ is uniform we have

$$g_{eff}(s) = \left\{ \begin{array}{l} 0 \text{ for } k-s \leq -\sqrt{3}\sigma_\eta \\ \frac{1}{2}\left(1+\frac{k-s}{\sqrt{3}\sigma_\eta}\right) \text{ for } |k-s| < \sqrt{3}\sigma_\eta \\ 1 \text{ for } k-s \geq \sqrt{3}\sigma_\eta \end{array} \right. . \quad (8)$$

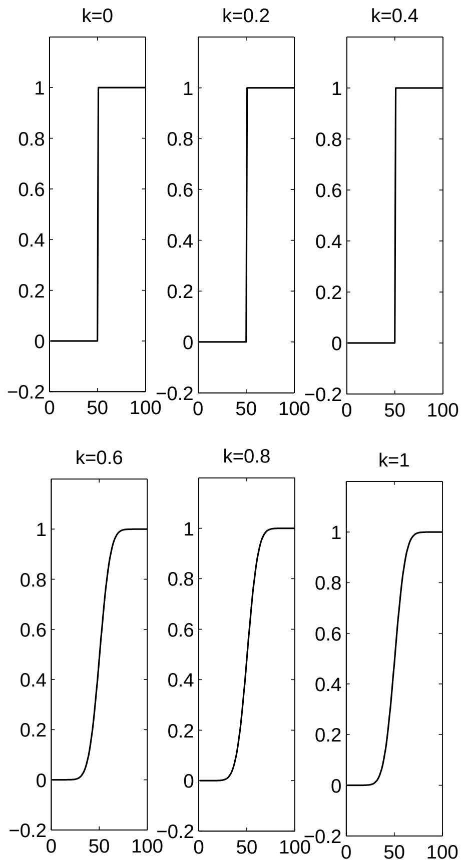

Figure 2. Illustration of the lack of robustness of classical Perona-Malik process regarding parameter k. $g(\cdot)$ is given by Eq. (5), iteration number n is fixed to 50, and time step τ to 0.2.

$g_{eff}(\cdot)$ function corresponds to the average theoretical equivalent characteristic of $g_\eta(\cdot)$ in presence of a purposely added noise with standard deviation σ_η.

B. Experiment

We now propose to compare the behavior of the numerical diffusion process of Eqs. (3) and (5) with the equivalent theoretical input–output characteristic of Eq. (6). We choose the noisy step ψ_0 of Fig. 3.(a), and we assess the efficacy of the restoration process with the normalized cross-covariance given by

$$C_{\psi_{ori}\psi(t_n)} = \frac{\langle(\psi_{ori} - \langle\psi_{ori}\rangle)(\psi(t_n) - \langle\psi(t_n)\rangle)\rangle}{\sqrt{\langle(\psi_{ori} - \langle\psi_{ori}\rangle)^2\rangle\langle(\psi(t_n) - \langle\psi(t_n)\rangle)^2\rangle}} \quad , \tag{9}$$

with $\langle..\rangle$ a spatial average, $\psi(t_n)$ the different restored steps calculated with Eq. (2), for (i) g_{eff} and (ii) g_η, at discrete instants $t_n = n\tau$. We are now in position to perform subjective and quantitative comparison of both $g_{eff}(\cdot)$ and $g_\eta(\cdot)$ functions.

As noticeable on Figs. 3.(b) and 3.(c), restoration results are perfectly matching between numerical simulation and theoretical relation (standard deviation of ξ noise is set to 0.05 for illustration) .

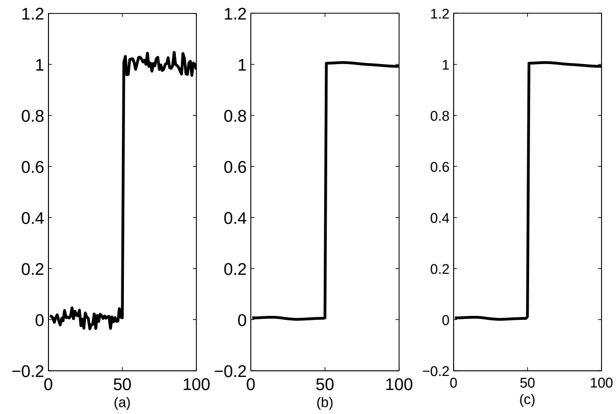

Figure 3. Comparison between real stochastic diffusion process (Eqs. (3) and (5) and theoretical one (Eq. (6)) on noisy step $\psi_0.\xi$ noise is gaussian of standard deviation fixed to 0.05. iteration number n is fixed to 150. (a) ψ_0, (b) noise-enhanced diffusion process, (c) diffusion process with $g(\cdot) = g_{eff}(\cdot)$.

This agreement is also valid in Fig. 4 which shows average evolution of normalized cross-covariance (Eq. (9)) in terms of iteration number n calculated for 1000 diffusion processes.

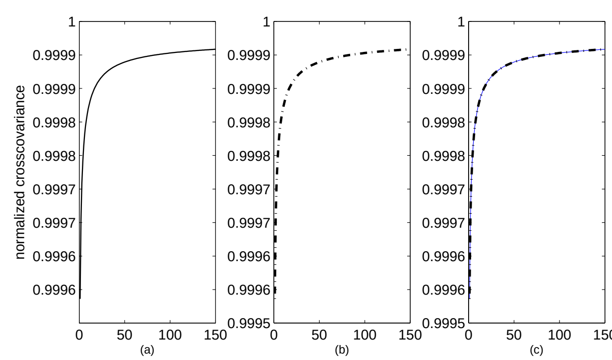

Figure 4. Comparison of evolution of normalized cross-covariance for 1000 diffusion processes (Eq. (9) between real stochastic diffusion process (Eqs. (3) and (5) and theoretical one (Eq. (6)) on noisy step ψ_0. n is fixed to 150. (a) noise-enhanced diffusion process, (b) diffusion process with $g(\cdot) = g_{eff}(\cdot)$, (c) superposition of both. One can notice that the scale for normalized crosscovariance is very tiny: this can be easily explained by the fact that even corrupted, the noisy version of the step function remains characterized by a high value of this parameter. Global variations still remain of primary importance and must be only considered for this study.

Fig. 4.(c) shows again a perfect matching between both average evolution curves.

These results establish the link between the useful-noise effect shown in [Histace and Rousseau, 2006] and the mechanism at work in static nonlinear systems as described in [Chapeau-Blondeau and Rousseau, 2002].

IV. Study of the Stochastic Resonance effect

In order to further study the influence of a purposely injection of noise in classical Perona-Malik process, we consider in this section that k in Eq. (5) is badly tuned (i.e. $k > 0.5$).

Considering the stochastic version of Perona-Malik process (Eq. (2)) with $g(\cdot)$ given by Eq. (5), the purposely injected noise η is a zero-mean Gaussian noise characterized by a tunable rms amplitude σ_η. For a visual appreciation of the noise-enhanced process, we consider the noisy step ψ_0 of Fig. 1 and k is set to 0.6, which corresponds to a badly tuned value regarding Fig. 2. In these conditions, as shown in Fig. 5.(b), Perona-Malik process fails in denoising ψ_0 without altering its integrity. If we now consider the stochastic Perona-Malik process of Eq. (2) with same parametrization of k, addition of noise η acts as a random resetting of parameter k, and, as shown in Fig. 5.(c), sometimes makes the preservation of the discontinuity of ψ_0 possible whereas k was badly tuned. It is important to notice, that this positive effect does not occur systematically, because of the random nature of the noise η.

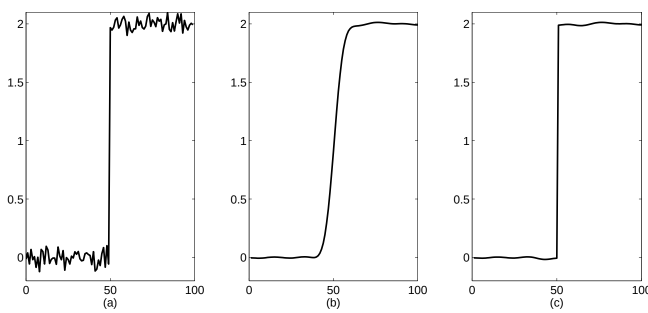

Figure 5. (a) Noisy step $\psi_0 = \psi_{ori} + \xi$ (rms amplitude of ξ is fixed to 0.05), (b) Perona-Malik restoration of ψ_0 ($n = 50$), (c) Stochastic Perona-Malik restoration of ψ_0 ($n = 50$ and $\sigma_\eta = 0.3$. For (b) and (c), k is fixed to 0.6 (badly tuned). Injection of η noise makes possible to obtain a better restoration of the noisy step regarding the fact that noise is suppressed and step discontinuity is preserved.

Although the positive effect of injection of η noise is not systematic, this clearly demonstrates that an increase of the robustness of classical Perona-Malik process regarding parameter k is possible with the function $g_\eta(\cdot)$ proposed. About the optimal amount of noise η to inject and about the possibility to estimate the probability to practically have a positive effect, we propose to quantitatively characterize the noise-enhanced effect shown Fig. 5.(c) with the calculation of the percentage of well-restored steps (no alteration of the discontinuity) among a large number N of restoration attempts, denoted N, and for different values of σ_η, k being set up to a non optimal value. This ratio can be interpreted as a measure of the gain of robustness compare to the classical Perona-Malik process of Eq. (1) toward threshold k. Fig. 6 shows the evolution of the percentage of well restored steps for $k = 0.6$.

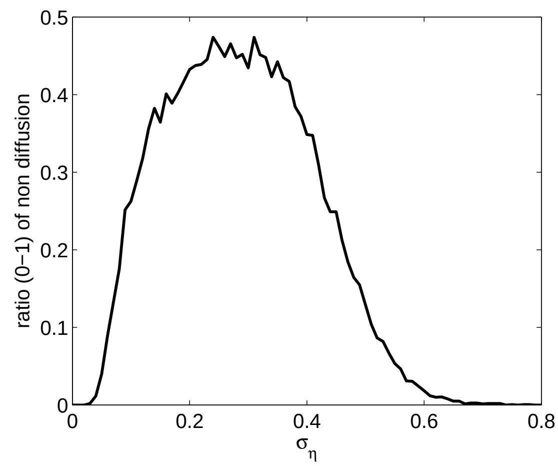

Figure 6. Variation of the ratio of well restored steps (no alteration of the discontinuity) thanks to the purposely injection of η (Eq. (2)) function of rms amplitude σ_η. k is fixed to 0.6 and N, the total amount of restoration attempts, to 1000.

First, one can notice that the variations of the ratio of well restored steps is typical of the existence of a stochastic resonance effect related to a static non linearity: we clearly notice in Fig. 6that the proposed ratio reach a maximum for a non zero amount of injected noise. Same experiments can be made for other badly-tuned values of k. Results are presented Fig. 7.

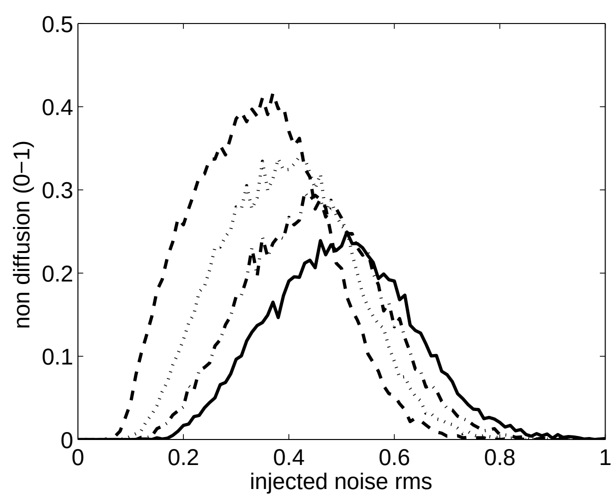

Figure 7. Ratio of non-diffused steps function of rms amplitude σ_η. N is fixed to 1000. Dashed line stands for $k = 0.65$, dotted one for $k = 0.7$, dash-dotted one for $k = 0.75$ and solid one for $k = 0.8$. For each value of k, same stochastic effect as before (Fig. 6) can be observed : the non-diffusion ratio is maximum for a non zero amount of purposely injected noise.

As visible in Fig. 7, even if the maximum value of the ratio decreases, the useful-noise effect can be observed. This decrease can be easily explained by the fact the farer parameter k is from 0.5, the more important is the necessary amount of noise to inject to finally make an interesting retuning of k. As a consequence positive effect of purposely injected noise η is less important and presents a maximum for a value of σ_η also increasing (which can also be noticed on Fig. 7). Moreover, that type of curves also makes an evaluation of the optimal amount of noise to add regarding k values. For instance, it appears that for $k = 0.6$ (Fig. 6), a maximum probability of 46% of non diffusion of the discontinuity of ψ_{ori} can be reached for $\sigma_\eta = 0.3$ thanks to the stochastic Perona-Malik process.

V. CONCLUSION

In this report, we have established a link between noised-enhanced anisotropic diffusion and stochastic resonance in static nonlinearities. This shows the way to non trivial transposition of 1D stochastic resonance effect to images. Further investigations in the continuity of this report could deal with extensions to more complex non-linear PDE of the literature.

More precisely, in some recent publications ([Morfu, 2009], [Histace and Ménard, 2010]) dealing with diffusion processes for image restoration, particular nonlinear anisotropic PDE, integrating a double-well potential function of the form $f(\psi) = \psi(\psi - a)(\psi - 1)$, have been proposed. One of the obtained PDE [Morfu, 2009] is an extension of the Fisher equation, derived from the Perona-Malik PDE, and given by

$$\frac{\partial \psi}{\partial t} = div(g(\|\nabla\psi\|)\nabla\psi) + f(\psi)\,. \tag{10}$$

Moreover, noticing that Eq. (11) can be related to the evolution equation of dynamic systems as described in [Chapeau-Blondeau, 2000] for instance, for which SR phenomenon have been clearly identified, a complete theoretical and practical study of those type of PDE could be of real interest for image restoration. For such a study, the considered image restoration PDE could be of the form

$$\frac{\partial \psi}{\partial t} = div(g(\|\nabla\psi\|)\nabla\psi) + f(\psi) + \eta(x,y)\,. \tag{11}$$

Establishment of a link between Fisher equation and stochastic resonance in dynamic nonlinearities could be of real interest to propose original restoration processes based on SR PDE and would complete the study proposed in this article where the focus was put only on static nonlinearities.

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